MS&E-345 Final Presentation
Pricing High Dimensional American Options

Ajaykumar Rajasekharan

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(Longstaff and Schwartz, 2001)

(Berridge and Schumacher, 2002,2004)

Contents

1 Motivation

- Have been increasingly important with the development of more and more complex contracts, for eg. valuations of basket options, swaptions etc.
- Traditional methods such as
  - Lattice Methods (e.g. Binomial Tree) (dynamic programming)
  - Finite Difference, Finite Element etc.. (solving complementarity problem)

  are good for early exercise computations but are limited by the number of stochastic factors $O(N^d)$, where, N is the number of grids (CURSE OF DIMENSIONALITY !!)

2 Simulation Based Methods

- Dynamic Programming, for $I_i = max\{K - S_i, 0\}$

$$V_i(S_i) = \max\left\{I_i(S_i), \mathrm{E}_i^Q\left[e^{-r\cdot\Delta t}V_{i+1}(S_{i+1})|S_i\right]\right\} \qquad (1)$$

- The optimal exercise strategy is determined by choosing between the current exercise value and the conditional expectation of the payoff from continuing to keep the option alive
- Direct generalization not feasible for multiple stochastic factors – again due to CURSE OF DIMENSIONALITY
- Need to find a good approximation of the expected value function $\mathrm{E}_i^Q\left[e^{-r\cdot\Delta t}V_{i+1}(S_{i+1})|S_i\right]$ – Provides Lower Bound

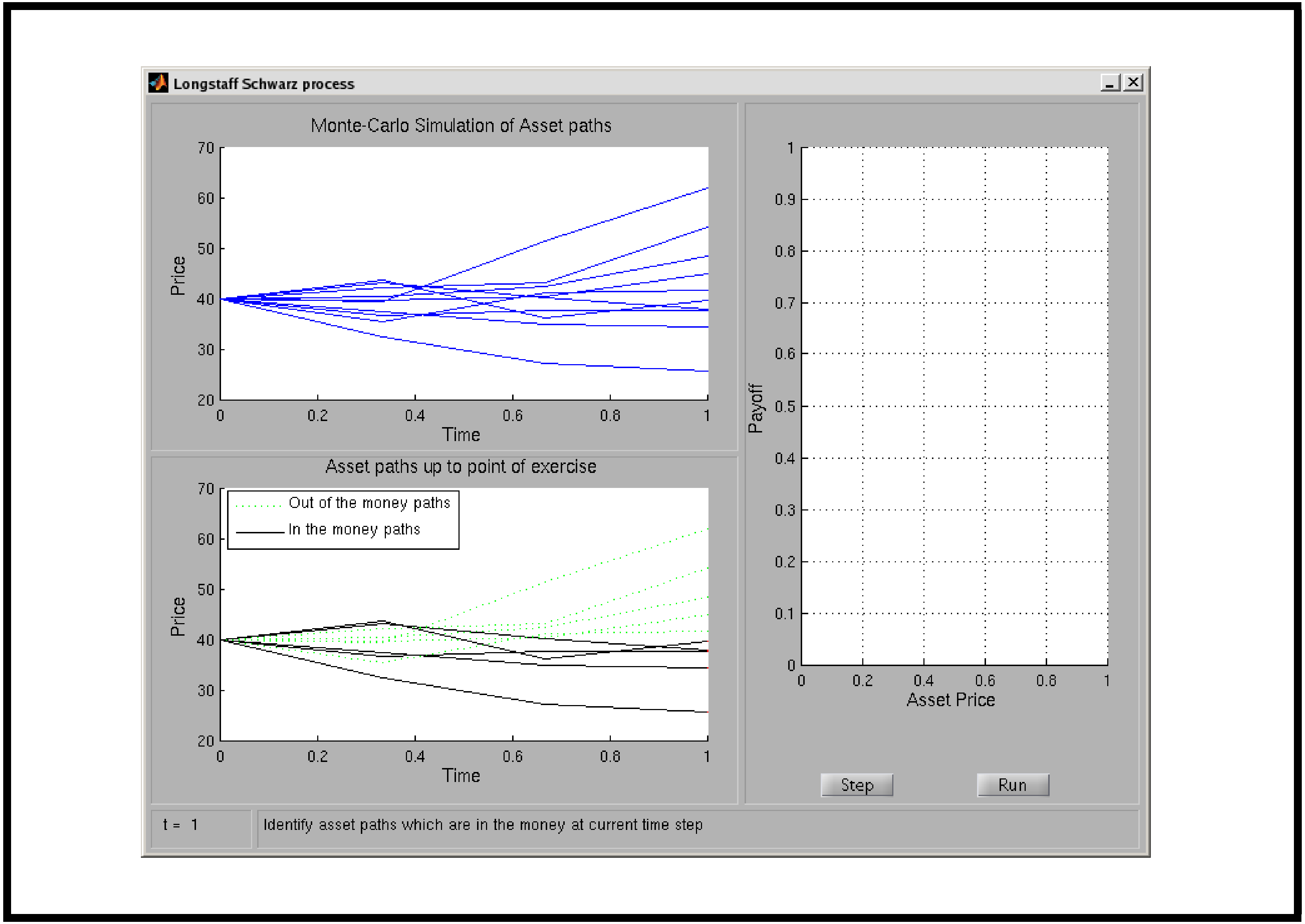
3 Least-Squares Method

- Longstaff & Schwartz, 2001

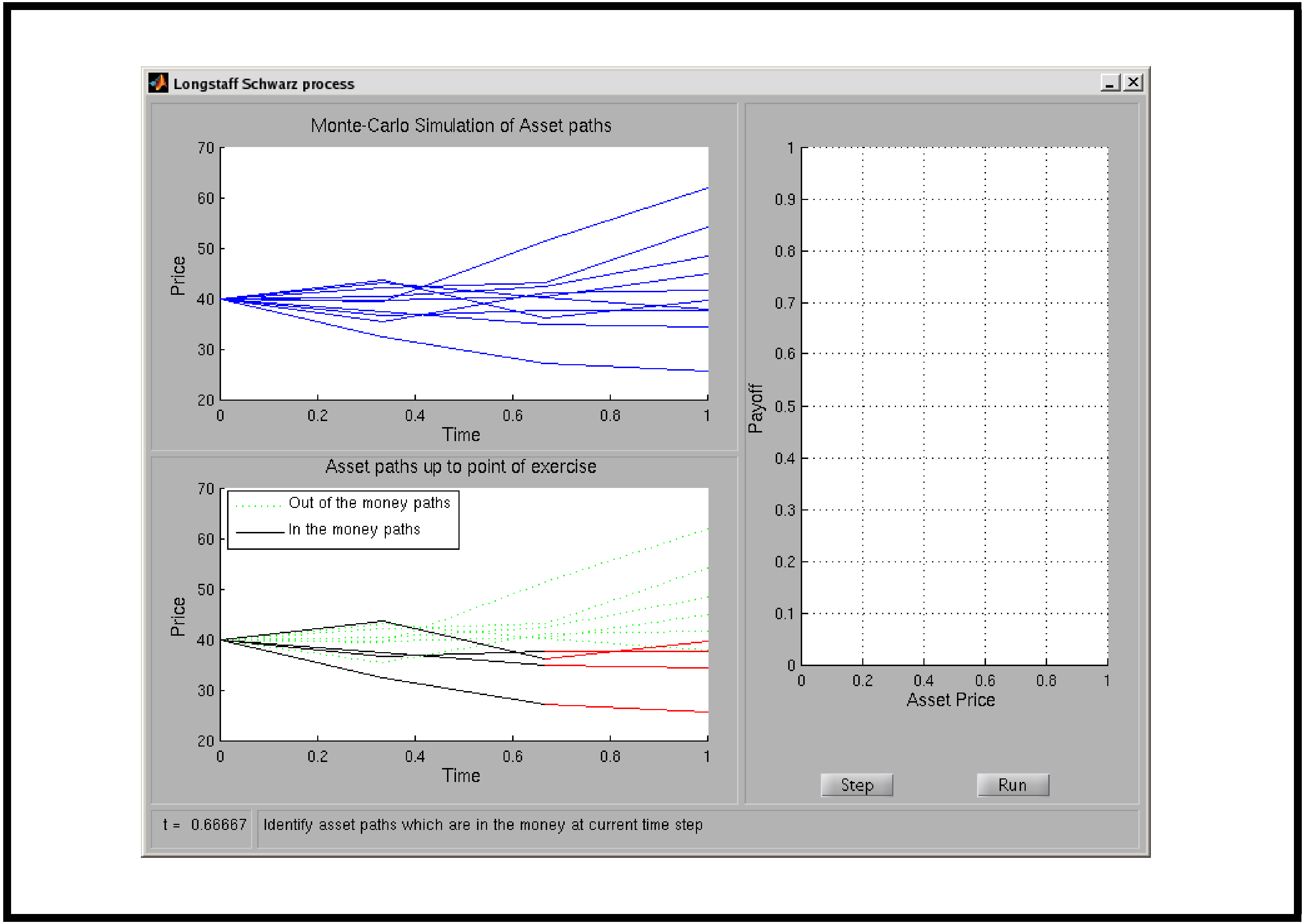
$$\mathrm{E}_i^Q\left[e^{-r\cdot\Delta t}V_{i+1}(S_{i+1})|S_i\right] \approx a_1 + a_2 S_i + a_3 S_i^2 \qquad (2)$$

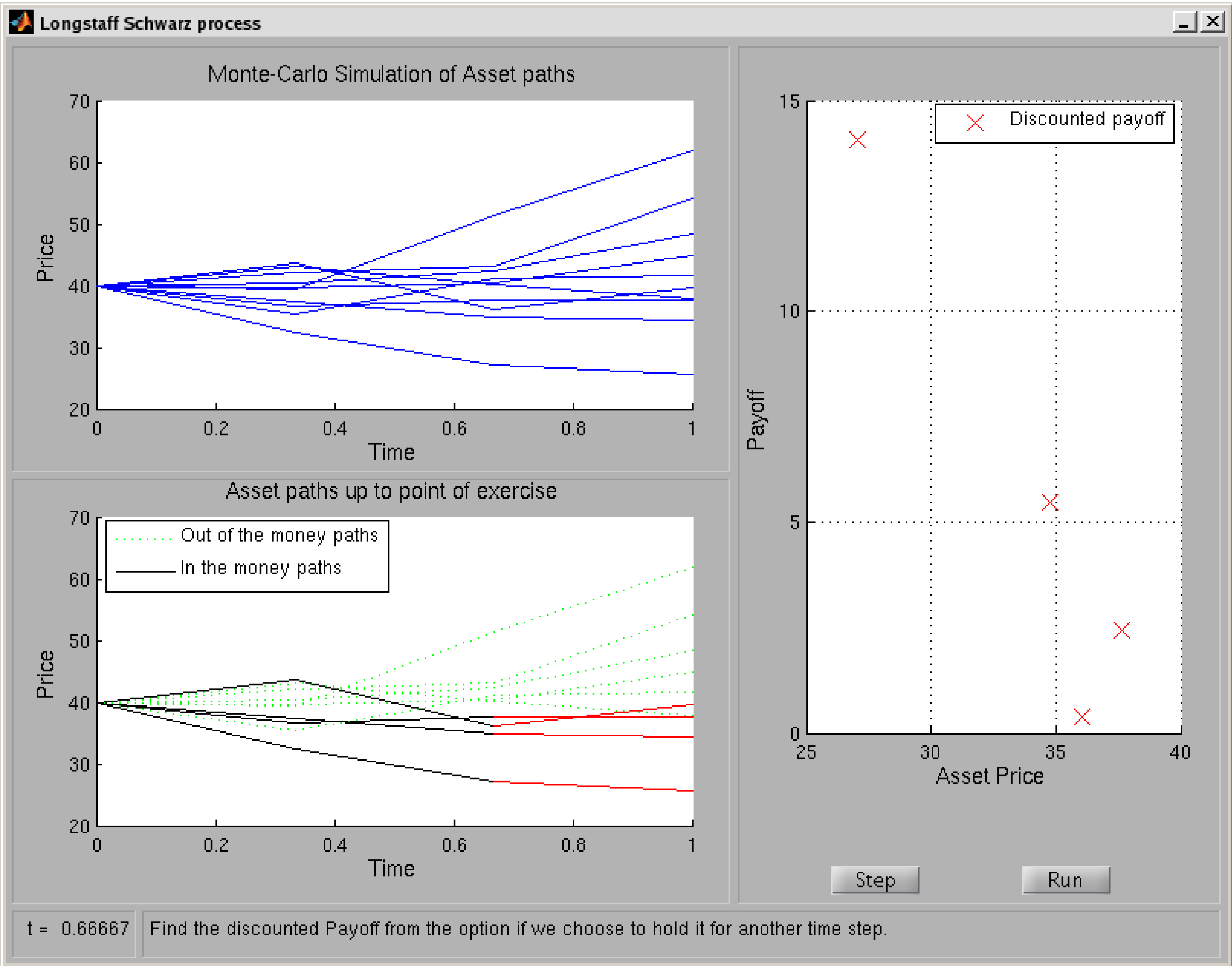
- The conditional expectation of the payoff from continuing to keep the option alive is estimated by regressing the subsequent realized cash flows from continuation on a set of basis functions of the values of the relevant state variables

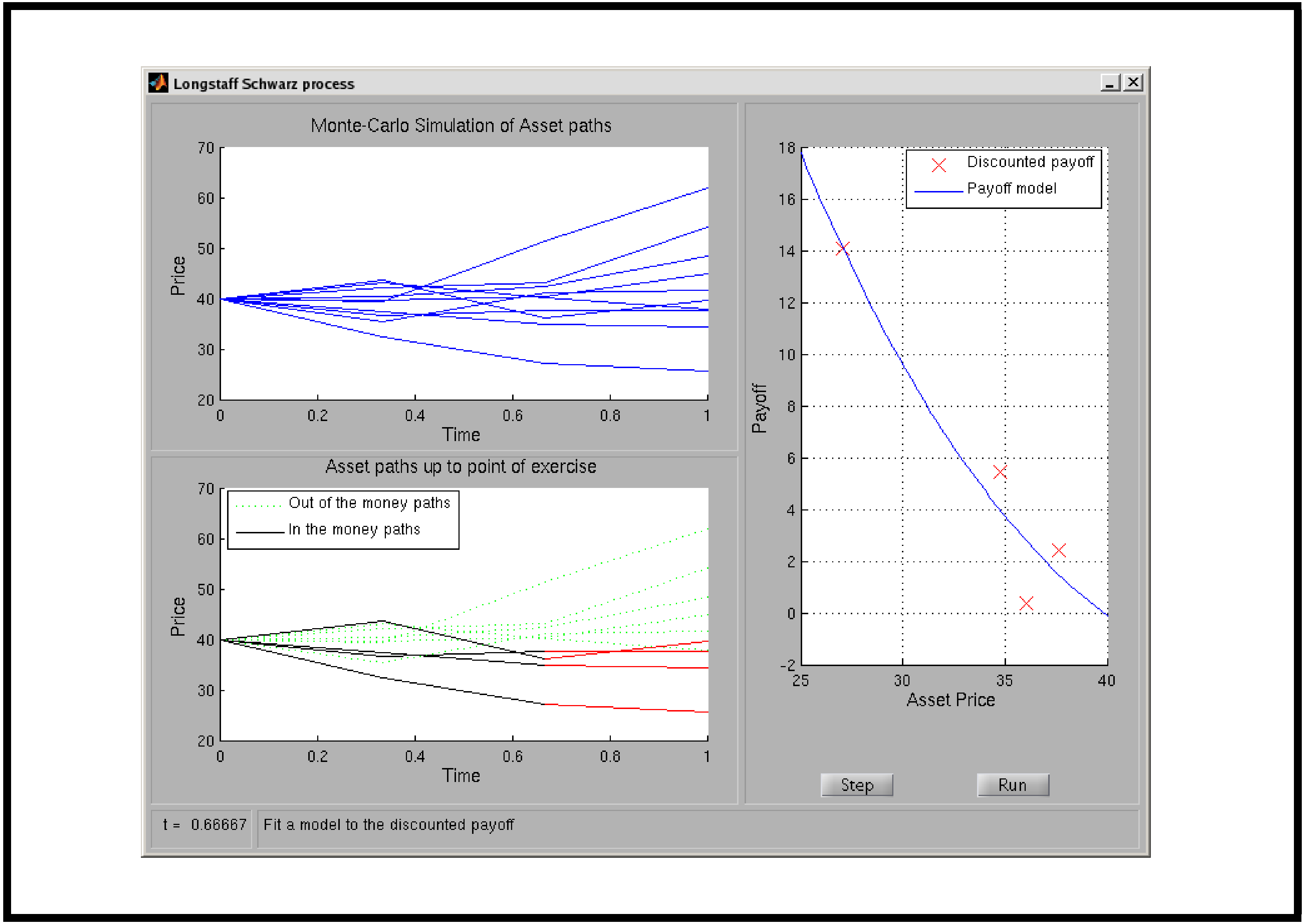
- A good choice of basis function unknown before hand

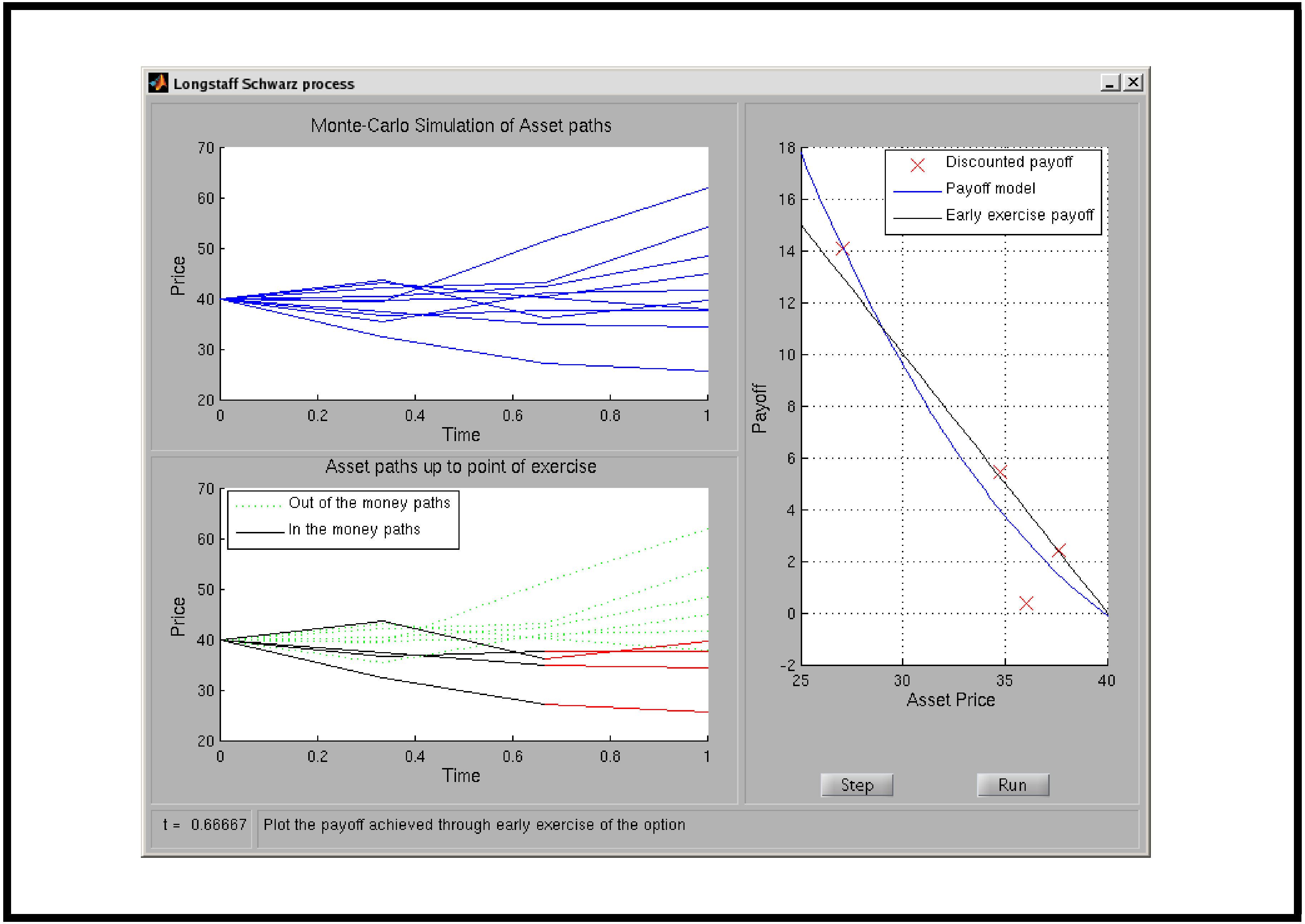


Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Price
Time
Payoff
Asset Price
Step
Run
t = 1
Identify asset paths which are in the money at current time step

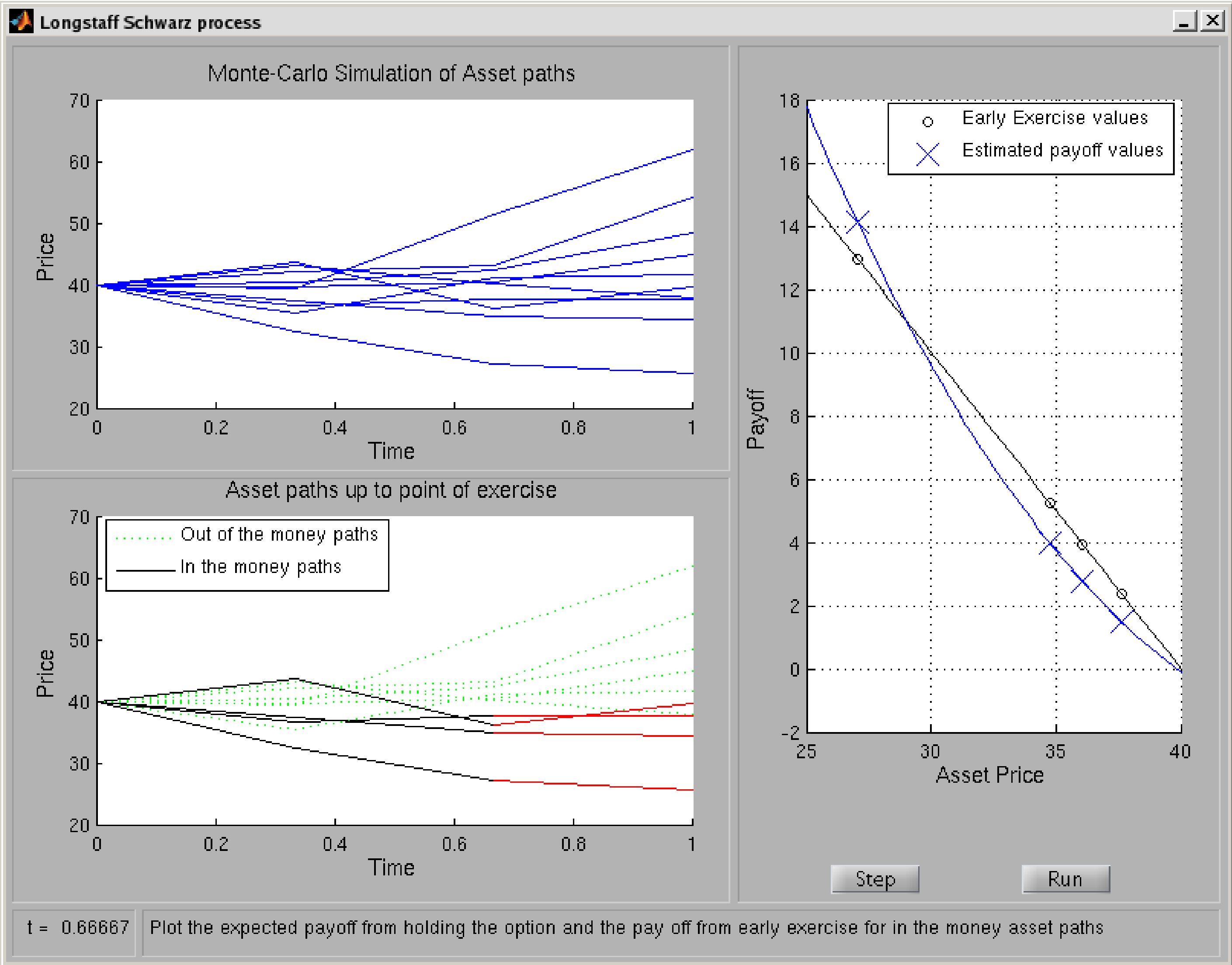

Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Payoff
Asset Price
Step
Run
t = 0.66667
Identify asset paths which are in the money at current time step

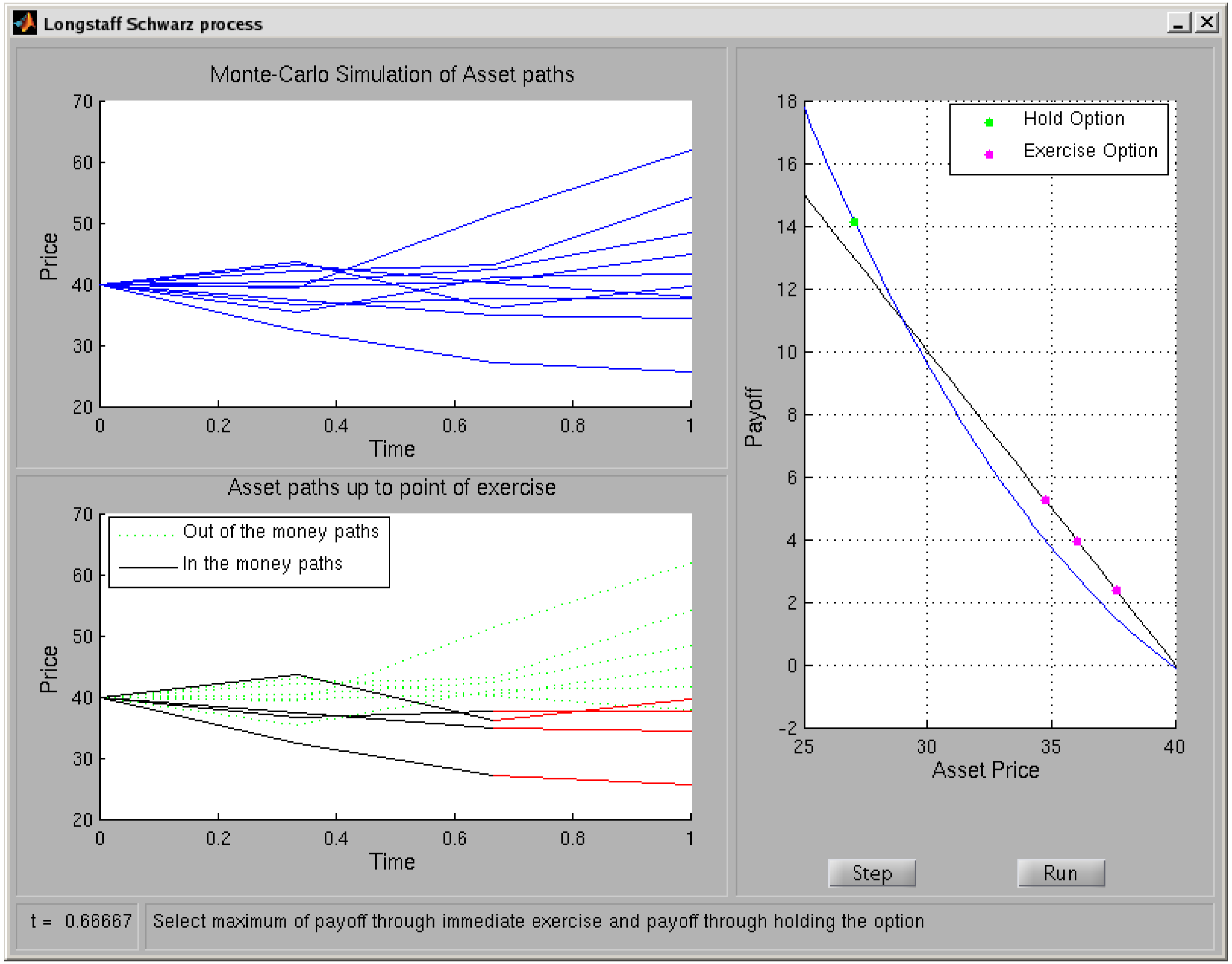

Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Discounted payoff
Payoff
Asset Price
Step
Run
t = 0.66667
Find the discounted Payoff from the option if we choose to hold it for another time step.

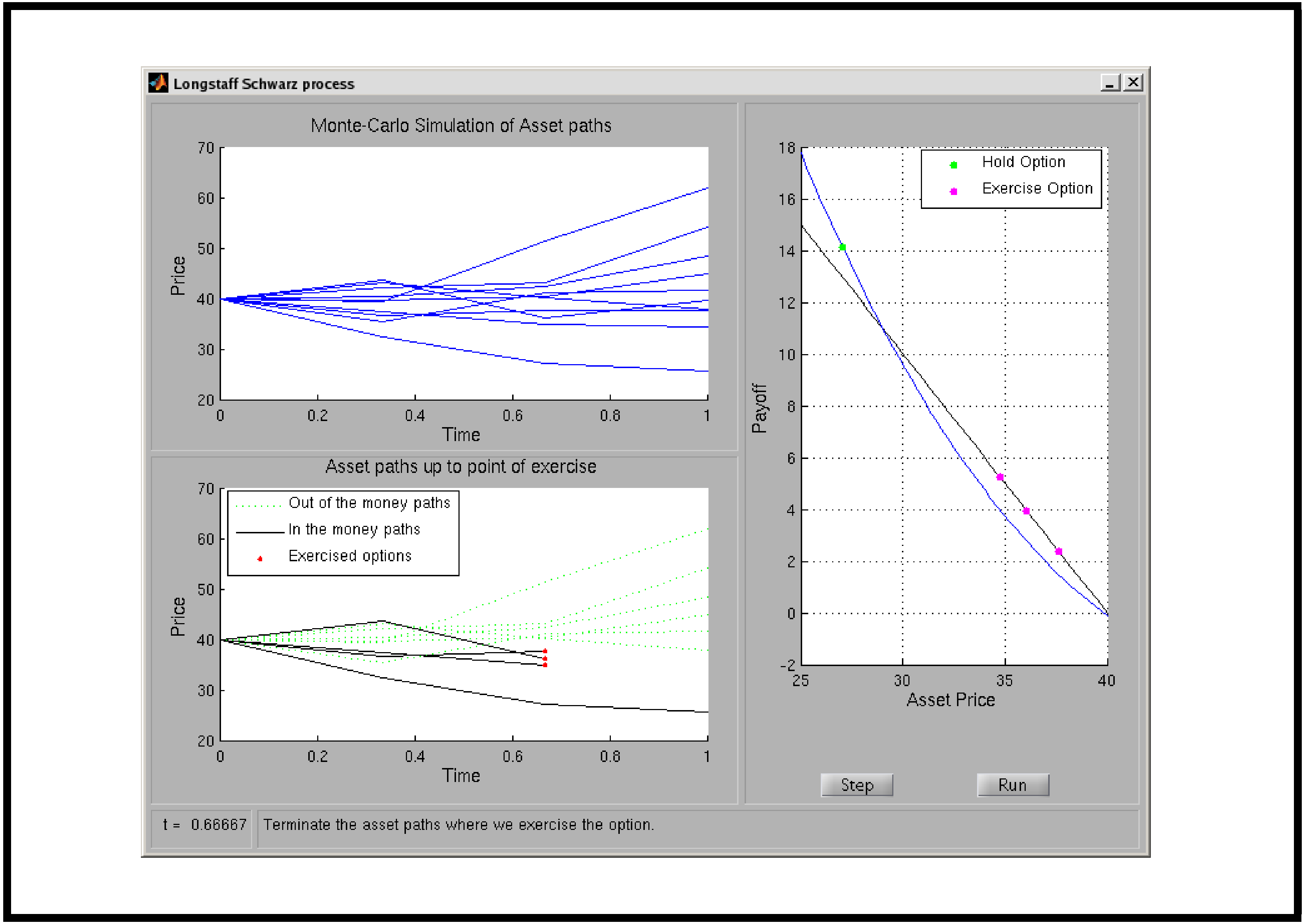

Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Payoff
Asset Price
Discounted payoff
Payoff model
Step
Run
t = 0.66667
Fit a model to the discounted payoff

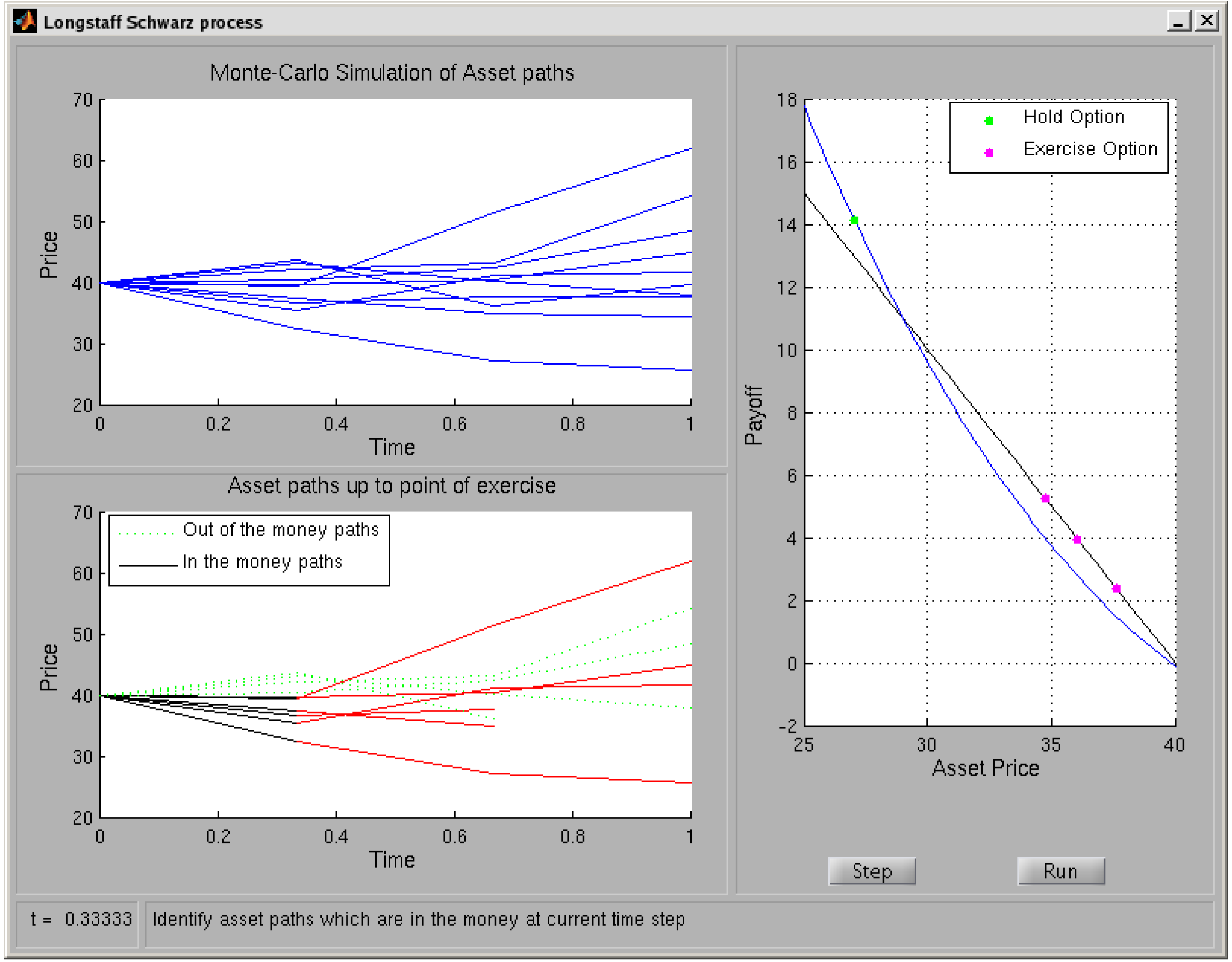


Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Price
Time
Discounted payoff
Payoff model
Early exercise payoff
Payoff
Asset Price
Step
Run
t = 0.66667
Plot the payoff achieved through early exercise of the option


Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Early Exercise values
Estimated payoff values
Payoff
Asset Price
Step
Run
t = 0.66667
Plot the expected payoff from holding the option and the pay off from early exercise for in the money asset paths


Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Price
Time
Hold Option
Exercise Option
Payoff
Asset Price
Step
Run
t = 0.66667
Select maximum of payoff through immediate exercise and payoff through holding the option

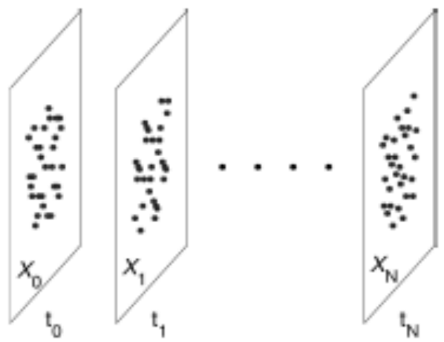

Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Exercised options
Hold Option
Exercise Option
Payoff
Asset Price
Step
Run
t = 0.66667
Terminate the asset paths where we exercise the option.

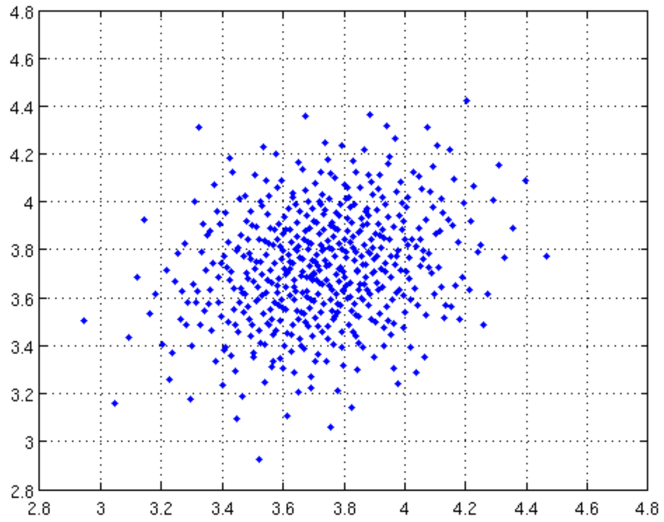

Longstaff Schwarz process
Monte-Carlo Simulation of Asset paths
Price
Time
Asset paths up to point of exercise
Out of the money paths
In the money paths
Hold Option
Exercise Option
Payoff
Asset Price
Step
Run
t = 0.33333
Identify asset paths which are in the money at current time step

- Advantages
  - Quick solution obtainable (runs in MATLAB with 100000 sample paths, 50 time slices and 5 dimensions in less than 5 minutes)
  - Easy to code
- Disadvantages
  - Choice of basis function is tricky
  - Can get into numerical issues with singularity
  - Gives always a lower bound, need to complement it with the dual Monte-Carlo approach (L.C.G. Rogers) that gives and upper bound

$$V_i(S_i) \quad = \quad \min \left\{ I_i(S_i), \mathrm{E}_i^Q \left[\max \left(e^{-r \cdot \Delta t} V_{i+1}(S_{i+1}) - M(s) \right) | S_i \right] \right\}$$

4 Irregular Grid Method





- Berridge and Schumacher, 2002,2004
- Closely related to the Method of Lines for solving PDE's

- The Complementarity Problem (see Willmott)

$$\begin{aligned} \frac{\partial v}{\partial t} + \mathcal{L}v &\leq 0 \\ v - \psi &\geq 0 \\ (\frac{\partial v}{\partial t} + \mathcal{L}v)'(v - \psi) &= 0 \end{aligned} \tag{3}$$

- Its Semi-Discrete Form

$$\begin{aligned} \frac{dv}{dt} + Av &\leq 0 \\ v - \psi &\geq 0 \\ (\frac{dv}{dt} + Av)'(v - \psi) &= 0 \end{aligned} \tag{4}$$

- Need for find the discrete infinitesimal generator A without doing differencing as in Finite-Difference

- A can be computed using the consistency conditions namely

$$\max f_j \cdot a_{i,j} \quad s.t. \tag{5}$$

$$\begin{aligned}
\Sigma(x_i) &= \sum_{j=1, j\neq i}^{n} (x_j - x_i)(x_j - x_i)' a_{i,j} \\
\mu_{RN} &= \sum_{j=1, j\neq i}^{n} (x_j - x_i) a_{i,j} \\
1 &= \sum_{j=1, j\neq i}^{n} a_{i,j} \\
a_{i,j} &\geq 0, \quad i \neq j \\
a_{i,i} &= -\sum_{j\neq i}^{n} a_{i,j}
\end{aligned}$$

where, $a_{i,j}$ is the (i, j)th entry of of the infinitesimal generator A.

Algorithm

- Choose grid size n
- Generate a constant QMC grid χ
- Compute generator matrix A
- Choose a time integrator (used second order implicit Crank-Nicholson)
- Solve the linear complementarity problem (PSOR technique)

- Advantages
  - Semi-discrete (finite-difference) style method – convergence with more grid points and higher order time integration
  - Optimum grid positioning unlike finite difference
  - Reusability of the infinitesimal generator A
- Disadvantages
  - The construction of grid is iffy
  - Handling boundary conditions (i.e. points where no feasible solution to the LP) is not clear (have to solve the LP to see if there are any infeasible points, can't determine before hand)

5 Numerical Experiments

- A geometric average put option written on d assets with the following payoff function is considered here

$$\psi(s) = \Big(K - \big(\prod s_i\big)^{1/d}\Big)^+ \tag{6}$$

- This option is equivalent to a standard put option on an asset with starting value $\exp \overline{X}_0$ (average log price), strike price K, risk free rate r and continuous dividend stream $\delta = \frac{1}{2}\Big(\frac{1}{d}\sum \sigma_i^2 - \overline{\sigma}^2\Big)$
- $S_0 = 40, K = 40, T = 1, r = 0.06, \sigma_{ii} = 0.4, \rho = 0.25$
- LSM : 100,000 sample paths, 50 time slices, cubic regression
- Irregular grid method : average of 10 QMC grids with 2000 points each, 50 time steps

Table 1: American Geometric Average Put Option

Dimension	Sol in Paper	Longstaff-Schwarz	Berridge-Schumacher
2	1.7787	1.6051	
		(0.0122)	
3	1.5597	1.3903	1.6412
		(0.0106)	(0.0072)
4	1.4392	1.2721	1.4067
		(0.0098)	(0.0255)
5	1.3625	1.1946	1.1305
		(0.0093)	(0.0397)

6 Conclusions

- Looked a two methods – Least-Squares Method and the Irregular Grid Method
- Both have advantages and disadvantages as described before
- Didn't obtain good results with Irregular grid method, probably due to bug, or due to the non-optimal solution of LP with MATLAB or some innate features in the grid generation process or incorrect handling of boundary conditions